Fixing the impulse equations in Inhwan Han '05 "*Dynamics in Carom and Three Cushion Billiards*"

So basically I enjoy the treatment of ball-cushion collisions here as it is simple, analytic, and appears to be physically plausible, unlike the blatant flaws in Marlow's treatment. But some of the math was not adding up. So jumping right into it, if you assume a perpendicular collision with rail heights equal to the ball radius and no spin, you get the following initial condition:

$v_{X0} = 1$
$v_{Y0} = 0$
$\theta_c = 0$
$\theta_0 = 0$

I chose this simple scenario to test whether or not Han's equations yield the correct outgoing velocity, $v_X' = -e\, v_{X0}$

Indeed, Eq 24 yields for this scenario:

$$v_X' = v_{X0} - v_{X0}\cdot\left[\frac{2}{7}\cdot\sin^2(\theta_a) + (1+e)\cdot\cos^2(\theta_a)\right] - \frac{2}{7}\cdot R\,\omega_{Y0}\sin(\theta_a)\Big]$$
$$= v_{X0} - v_{X0}\cdot(1+e)$$
$$= -e\, v_{X0}$$

This is a good start... But I am troubled because Eq 22 yields an impulse in the X direction of 0 for this case. So he basically has the correct final form, but the wrong intermediates. The problem is that I need the intermediates to solve for the angular velocity evolution (only the linear velocities are explictly stated). So to address this problem I have reverse engineered the equations to find out what the intermediates _must_ be to produce the presumably correct Eq 24a. Here are the correct values (modifications highlighted in red) :

Equations 14:

$sx := vX\cdot\sin(\text{theta}) + R\cdot wY$

$$vX\sin(\theta) + R\,wY \tag{1}$$

$sy := -vY - R\cdot wZ\cdot\cos(\text{theta}) + R\cdot wX\cdot\sin(\text{theta})$

$$-vY - R\,wZ\cos(\theta) + R\,wX\sin(\theta) \tag{2}$$

$c := + vX\cdot\cos(\text{theta})$

$$vX\cos(\theta) \tag{3}$$

Equations 15:

$A := \frac{7}{2\cdot m}$

$$\frac{7}{2\,m} \tag{4}$$

$B := \frac{1}{m}$

$$\frac{1}{m} \tag{5}$$

Equation 21a

$PX := -\frac{sx}{A}\cdot\sin(\text{theta}) - \frac{(1+e)\cdot c}{B}\cdot\cos(\text{theta})$

$$-\frac{2}{7}\left(vX\sin(\theta) + R\,wY\right) m\sin(\theta) - (1+e)\,vX\cos(\theta)^2\,m \tag{6}$$

By plugging this into Eq 23, we retrieve the *presumably* correct form, i.e. Eq 24:

$vXp := vX + \frac{PX}{m}$

$$vX + \frac{-\frac{2}{7}\left(vX\sin(\theta) + R\,wY\right) m\sin(\theta) - (1+e)\,vX\cos(\theta)^2\,m}{m}$$

$simplify(vXp, size)$

$$-(1+e)\,vX\cos(\theta)^2 + vX - \frac{2}{7}\,vX\sin(\theta)^2 - \frac{2}{7}\sin(\theta)\,R\,wY \tag{8}$$

Similarly, Eq 24b can be recapitulated with no further modifications:

$PY := \frac{sy}{A}$

$$\frac{2}{7}\left(-vY - R\,wZ\cos(\theta) + R\,wX\sin(\theta)\right) m \tag{9}$$

$vYp := vY + \frac{PY}{m}$

$$\frac{5}{7}\,vY - \frac{2}{7}\,R\,wZ\cos(\theta) + \frac{2}{7}\,R\,wX\sin(\theta) \tag{10}$$

The only other sub-equation which has not been tested for a sanity check in Eq 21 is PZ, and I don't have a great test for it except that when $\theta_c = 0$ there is only a contribution from sx, which makes sense geometrically.

Ok, so that is Eq 21. What about Eq 22?

$PX2 := -\frac{\text{mu}\cdot(1+e)\cdot c}{B}\cdot\cos(\text{phi})\cdot\sin(\text{theta}) - \frac{(1+e)\cdot c}{B}\cdot\cos(\text{theta})$

$$-\mu\,(1+e)\,vX\cos(\theta)\,m\cos(\phi)\sin(\theta) - (1+e)\,vX\cos(\theta)^2\,m \tag{11}$$

$$vX2p := vX + \frac{PX2}{m}$$

$$vX + \frac{-\mu\,(1+e)\,vX\cos(\theta)\,m\cos(\phi)\,\sin(\theta) - (1+e)\,vX\cos(\theta)^2\,m}{m} \tag{12}$$

$simplify(vX2p, size)$

$$-\left(-1 + (1+e)\cos(\theta)^2 + \mu\cos(\phi)\sin(\theta)\,(1+e)\cos(\theta)\right) vX \tag{13}$$

Upon some manual work, you can massage this into this, which is Equation 25a:

$vX - vX\,(1+e)\,\cos(\mathrm{theta})\,[\cos(\mathrm{theta}) + \mathrm{mu}\cos(\mathrm{phi})\,\sin(\mathrm{theta})]$

$$vX - vX\,(1+e)\cos(\theta)\,[\cos(\theta) + \mu\cos(\phi)\sin(\theta)] \tag{14}$$

So that's good. Let's see if we can recapitulate Equation 25b:

$$PY2 := \frac{\mathrm{mu}\cdot(1+e)\cdot c}{B}\cdot\sin(\mathrm{phi})$$

$$\mu\,(1+e)\,vX\cos(\theta)\,m\sin(\phi) \tag{15}$$

$$vY2p := vY + \frac{PY2}{m}$$

$$vY + \mu\,(1+e)\,vX\cos(\theta)\sin(\phi) \tag{16}$$

Looks good.